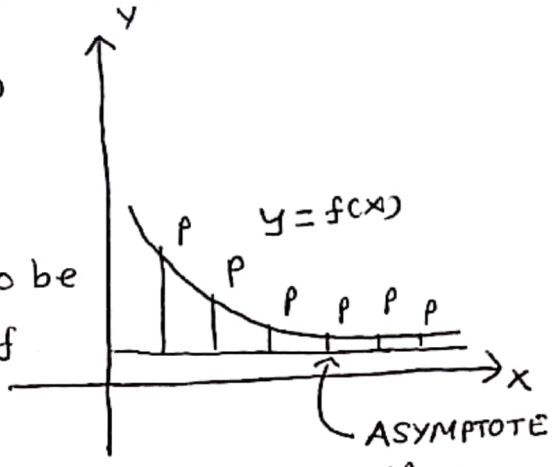
ASYMPTOTES



Let the ~~point~~ curve be $y = f(x)$

Let P be any point on this curve.

A straight line is said to be an asymptote of this curve if the point P moves towards infinity the perpendicular distance from the point P to the straight line tends to zero.

Types of asymptotes

There are two types of asymptotes

(i) Parallel asymptote

(ii) Oblique asymptote.

Parallel asymptote

Parallel asymptote are of two types

(a) Asymptote parallel to x-axis's

(b) Asymptote parallel to y-axis's.

(a) Asymptote parallel to x-axis

which can be obtained by equating the co-efficient of highest degree term of x to zero.

(b) Asymptote parallel to y-axis

which can be obtained by equating the co-efficient of highest degree term of y to zero.

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NOTE:- If the co-efficients are constants then there is no asymptotes parallel to co-ordinate axis's.

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Find the asymptote parallel to the co-ordinate axis.

1. $x^3+3xy^2+y^2+2x+y=0$

Ans:- The co-efficient of highest degree term of x is 1 which is a constant. So there is no asymptote parallel to x-axis.

The co-efficient of highest degree term of y is $3x+1$. So the asymptote parallel to y-axis is

$$3x+1=0$$

3. $x^2y^3+x^3y^2=x^3+y^3$

$\Rightarrow x^2y^3+x^3y^2-x^3-y^3=0$

The co-efficient of highest degree term of x is y^2-1. So, the Asymptote parallel to x-axis is

$$y^2-1=0$$

$$\Rightarrow y=\pm 1.$$

The co-efficient of highest degree term of y is x^2-1. So, the asymptote parallel to y-axis is

$$x^2-1=0$$

$$\Rightarrow x=\pm 1.$$

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Oblique Asymptote

Example:-1

Find the asymptotes of

$$y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$$

Ans:-

The co-efficients of highest degree term of x and y are constants.

So, there is no asymptote parallel to the co-ordinate axis.

Put $x = 1$ and $y = m$ in the 3rd and 2nd degree term of the equation

$$\varphi_3(m) : m^3 - m - 2m^2 + 2$$

$$\varphi_2(m) : -7m + 3m^2 + 2$$

To find m put $\varphi_3(m) = 0$

$$\Rightarrow m^3 - m - 2m^2 + 2 = 0$$

$$\Rightarrow m(m^2 - 1) - 2(m^2 - 1) = 0$$

$$\Rightarrow (m^2 - 1)(m - 2) = 0$$

$$\Rightarrow (m + 1)(m - 1)(m - 2) = 0$$

$$\Rightarrow m = -1, 1, 2$$

To find m

Put $\varphi_n(m) = 0$ where n is the highest degree

C can be given by the formula

$$C = -\frac{\varphi_2(m)}{\varphi_3'(m)}$$

$$= -\left[\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}\right]$$

To find C

C can be given by the formula

$$C = -\frac{\varphi_{n-1}(m)}{\varphi_n'(m)}$$

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When $m = -1$, $c = -2$

$m = 1$, $c = -1$

$m = 2$, $c = 0$

So, the asymptotes are

$y = -1 \cdot x - 2 \Rightarrow x + y + 2 = 0$

$y = 1 \cdot x - 1 \Rightarrow y - x + 1 = 0$

$y = 2 \cdot x + 0 \Rightarrow y - 2x = 0$

Exercise

Find the asymptotes of the following curves

1. $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 = 1$
2. $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0$
3. $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y = 3$

Exceptional case (Repeated values of m)

Example-2

Find the asymptotes of

$$y^3 + x^2y + 2xy^2 - y + 1 = 0$$

Ans:- The co-efficient of highest degree term of x is y.

So, the asymptote parallel to x-axis is $y = 0$

The co-efficient of highest degree term of y is 1. which is a constant. So there is no asymptote parallel to y-axis.

Put $x = 1$ and $y = m$ in the 3rd and 2nd degree term of the equation

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$\phi_3(m) : m^3 + m + 2m^2$

$\phi_2(m) : 0$

To find m put

$$\phi_3(m) = 0$$

$$\Rightarrow m^3 + 2m^2 + m = 0$$

$$\Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m(m+1)^2 = 0$$

$$\Rightarrow m = 0, -1, -1$$

C can be given by the formula

$$C = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{0}{3m^2 + 4m + 1}$$

When, $m = 0$, $C = \frac{0}{1} = 0$ Thus the asymptote is given by

$$y = 0 \cdot x + 0 \Rightarrow y = 0$$

When $m = -1$, C is undefined

For repeated value of m C can be given by the formula

$$\frac{1}{2} C^2 \phi_3''(m) + C\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \frac{1}{2} C^2 (6m + 4) + C(0) - m = 0$$

When $C = -1$,

$$-C^2 + 1 = 0$$

$$\Rightarrow C^2 = 1 \Rightarrow C = \pm 1$$

So, the asymptotes are

$y = -1 \cdot x + 1 \Rightarrow y + x - 1 = 0$

$y = 1 \cdot x - 1 \Rightarrow y - x + 1 = 0$

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NOTE:-

For repeated value of m, c can be given by the formula

$$\frac{1}{2} c^2 \phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

where n is the highest degree.

Exceraise

Find all asymptotes of

1. $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$
2. $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 = 2$
3. $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 = 1$

Another method for finding out the oblique asymptotes for less number of terms

Example-4

Find all asymptotes of

$$x^3 + y^3 - 3xy = 0$$

Ans:- There is no asymptotes parallel to co-ordinate axis.

Put $y = mx + c$

$$x^3 + (mx+c)^3 - 3x(mx+c) = 0$$

$$\Rightarrow x^3 + m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3 - 3mx^2 - 3cx = 0$$

$$\Rightarrow (1+m^3)x^3 + (3m^2c - 3m)x^2 + (3mc^2 - 3c)x + c^3 = 0$$

Equating both sides,

$$1 + m^3 = 0 \Rightarrow m^3 = -1 \Rightarrow m = -1$$

$$3m^2c - 3m = 0 \Rightarrow c = \frac{3m}{3m^2} = \frac{1}{m}$$

when $m = -1$, $c = -1$ So, the asymptote is $y = -1x - 1$

$\Rightarrow x + y + 1 = 0$

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Exercise

Find all asymptotes of

(a) $y^3 = x^2(x-a)$ (b) $y^2(x^2-a^2) = x$

(c) $y^2(x-b) = x^3+a^3$ (d) $xy^2 = 4a^2(2a-x)$.

Asymptotes of non algebric curves

Example-1

Find the asymptote of $y = \tan x$

Ans:- we have, $y = \tan x$ ——— (1)

Slope $\frac{dy}{dx} = \sec^2 x$

Tangent at (x, y) is

$$Y - \tan x = (X - x)\sec^2 x$$

$$\Rightarrow Y\cos^2 x - \sin x \cos x = X - x \quad \text{——— (2)}$$

Now as $x \to \frac{\pi}{2}$, $y \to \infty$ in equation (1)

Taking limit as $x \to \frac{\pi}{2}$ in both sides of equation (2)

$$Y.0 - 1.0 = X - \frac{\pi}{2}$$

$\Rightarrow X = \frac{\pi}{2}$ is the required asymptote

The other asymptotes are

$$X = -\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$$

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Example :-2

Find the asymptotes of $y = \log x$

Ans:- We have,

$$y = \log x \quad \text{——— (1)}$$

Slope $\frac{dy}{dx} = \frac{1}{x}$

Hence, the equation of tangent at (x, y) is

$$y - \log x = \frac{1}{x}(x - x)$$

$$\Rightarrow xy - x\log x = x - x \quad \text{——— (2)}$$

In equation (1) $y \to \infty$ as $x \to 0$

Taking limit as $x \to 0$ in both sides of equation (2)

$$\lim_{x \to 0} (xy - x\log x) = \lim_{x \to 0} (x - x)$$

$$\Rightarrow 0 - 0 = x - 0$$

$\Rightarrow x = 0$ is the required asymptote.

Exercise

Find the asymptotes of

(i) $y = e^{-x^2}$ (ii) $y = e^{ax}$ (iii) $y = \sec x$

(iv) $y = \operatorname{cosec} x$

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Asymptotes of Polar curves

If α be a root of the equation $f(\theta)=0$ then

$$r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

is an asymptote of the curve

$$\frac{1}{r} = f(\theta)$$

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Asymptotes of Polar curves

If α be a root of the equation $f(\theta)=0$ then

$$\boxed{r\sin(\theta-\alpha)=\frac{1}{f'(\alpha)}}$$

is an asymptote of the curve

$$\frac{1}{r}=f(\theta)$$

Example

Find the asymptote of the curve

$$r\sin n\theta=a$$

Ans:-

We have,

$$r\sin n\theta=a$$

$$\Rightarrow \frac{1}{r}=\frac{\sin n\theta}{a}$$

Here, $f(\theta)=\frac{\sin n\theta}{a}$

Let $f(\theta)=0$

$$\Rightarrow \frac{\sin n\theta}{a}=0$$

$$\Rightarrow \sin n\theta=0$$

$$\Rightarrow \sin n\theta=\sin m\pi$$

$$\Rightarrow n\theta=m\pi$$

$$\Rightarrow \theta=\frac{m\pi}{n}=\alpha \quad (\text{SAY})$$

$$f'(\theta)=\frac{n\cos n\theta}{a}$$

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$$f'(\alpha) = f'\left(\frac{m\pi}{n}\right) = \frac{n\cos n\frac{m\pi}{n}}{a} = \frac{n\cos m\pi}{a}$$

so, the required asymptote is

$$r\sin\left(\theta - \frac{m\pi}{n}\right) = \frac{1}{\left(\frac{n\cos m\pi}{a}\right)}$$

$$\Rightarrow r\sin\left(\theta - \frac{m\pi}{n}\right) = \frac{a}{n\cos m\pi}$$

Example

Find the asymptote of the curve

$$r\theta = a$$

Ans:. we have,

$$r\theta = a$$

$$\Rightarrow \frac{1}{r} = \frac{\theta}{a}$$

So, $f(\theta) = \frac{\theta}{a}$

Thus, $f(\theta) = 0$

$$\Rightarrow \frac{\theta}{a} = 0$$

$$\Rightarrow \theta = 0 = \alpha \text{ (SAY)}$$

$$f'(\theta) = \frac{1}{a}$$

$$f'(\alpha) = f'(0) = \frac{1}{a}$$

The required asymptote is

$$r\sin(\theta - 0) = \frac{1}{\left(\frac{1}{a}\right)}$$

$$\Rightarrow r\sin\theta = a$$

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Exercise

Find the asymptotes of

(i) $r \sin 2\theta = a$ (ii) $r = \frac{\pi}{\theta}$ (iii) $r \cos\theta = a \sin\theta$

(iv) $r = a \operatorname{cosec}\theta + b$.

Circular asymptotes

If the equation to a curve is $r = f(\theta)$ and if $\lim_{\theta \to \infty} f(\theta) = a$.

Then $r = a$ is called the required circular asymptote.

Example

Find the circular asymptote of the curve, $r = \frac{\theta^3 - 1}{\theta^3 + 1}$

Ans: we have, $r = \frac{\theta^3 - 1}{\theta^3 + 1}$

Here, $f(\theta) = \frac{\theta^3 - 1}{\theta^3 + 1}$

$$\lim_{\theta \to \infty} f(\theta) = \lim_{\theta \to \infty} \frac{\theta^3 - 1}{\theta^3 + 1} = \lim_{\theta \to \infty} \frac{1 - \frac{1}{\theta^3}}{1 + \frac{1}{\theta^3}}$$

$$= 1.$$

So, $r = 1$ is the required circular asymptote.

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Example

Find the asymptote of the curve

$$r(\theta^2+1) = \theta^2-1$$

Ans:- We have,

$$r(\theta^2+1) = \theta^2-1$$

$$\Rightarrow r = \frac{\theta^2-1}{\theta^2+1}$$

Here, $f(\theta) = \frac{\theta^2-1}{\theta^2+1}$

$$\lim_{\theta\to\infty} f(\theta) = \lim_{\theta\to\infty} \frac{\theta^2-1}{\theta^2+1} = \lim_{\theta\to\infty} \frac{1-\frac{1}{\theta^2}}{1+\frac{1}{\theta^2}} = 1.$$

So, $r=1$ is the required circular asymptote.

Exercise

Find the circular asymptote of

$$r = (3\theta^2+2\theta+1) / (\theta^2+\theta+1).$$

Intersection of curve and asymptote.

NOTE:- A curve of degree n can have atmost n asymptotes.

An asymptote of a curve of degree n cuts the curve in $n-2$ points

Thus the n asymptote of a curve of degree n cuts the curve in $n(n-2)$ points.

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Example:-1

show that the asymptotes of the cubic $2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$ cut the curve in three points. The points of intersection of the curve & its asymptotes lie on the straight line $8x + 2y + 1 = 0$.

Ans:-

we have,

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$

There is no asymptotes parallel to the co-ordinate axis's.

Put $x = 1$ & $y = m$ in the 3rd & 2nd degree term of the equation

$$\varphi_3(m): 2m^3 - 2m - 4m^2 + 4$$

$$\varphi_2(m): -14m + 6m^2 + 4$$

To find m,

Put $\varphi_3(m) = 0$

$$\Rightarrow 2m^3 - 2m - 4m^2 + 4 = 0$$

$$\Rightarrow 2m(m^2 - 1) - 4(m^2 - 1) = 0$$

$$\Rightarrow (m^2 - 1)(2m - 4) = 0$$

$$\Rightarrow (m + 1)(m - 1)(2m - 4) = 0$$

$$\Rightarrow m = -1, 1, 2.$$

'c' can be given by the formula

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$$c = -\frac{\phi_2(m)}{\phi_3'(m)}$$

$$= -\left[\frac{6m^2 - 14m + 4}{6m^2 - 8m - 2}\right]$$

when $m = -1, \quad c = -2$

$m = 1, \quad c = -1$

$m = 2, \quad c = 0$

So, the asymptotes are $y = -1 \cdot x - 2 \Rightarrow y + x + 2 = 0$

$y = 1 \cdot x - 1 \Rightarrow y - x + 1 = 0$

$y = 2 \cdot x + 0 \Rightarrow y - 2x = 0$

So, the three asymptotes of the curve cuts the curve in $3(3-2) = 3$ points

The joint equation of the asymptotes is

$$(y + x + 2)(y - x + 1)(y - 2x = 0$$

$$\Rightarrow y^3 - x^2 y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2y - 4x = 0$$

Multiplying 2 both sides & subtracting from the equation of the curve we get

$$8x + 2y + 1 = 0$$

on which the points of intersection must lie.

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Exercise

1. show that the asymptotes of the cubic

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$

Cut the curve in three points which lie on the straight line $x - y + 1 = 0$

2. Show that the eight points of intersection of the curve

$$xy(x^2 - y^2) + x^2 + y^2 = a^2$$

& the asymptotes lie on a circle whose centre is at the origin.

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